# $\mathcal{H}_\infty$ Controller Design for a Multi-Agent System Based on a Replicated Control Structure

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***Abstract*— Several recent design techniques address the problem of distributed controller synthesis for multi-agent systems (MAS). However, since these methods use a decomposition based approach to reduce the complexity of the MAS, they cannot provide a guarantee of performance with the designed controller, and furthermore lead to conservative designs due to the Lyapunov shaping paradigm. This paper proposes a direct gradient-based-optimization synthesis technique for a MAS with fixed communication topology, exploiting the fact that the same controller is replicated on each agent. Although this results in a non-convex design problem, an appropriate initialization step makes the synthesis efficient and helps to obtain controllers with guaranteed performance on the MAS.**

## I. INTRODUCTION

This work considers the problem of controlling a multi-agent system (MAS), consisting of $N$ identical agents, that communicate with each other and have a common goal. There are many applications envisioned for such systems (car platoons, surveillance robots, satellite imaging formations, etc. [1], [2]). Whereas central controllers can be designed for such multi-agent systems, they are not practical due to the increased communication overhead and the reduced flexibility to changes.

The stability and performance analysis, as well as the controller synthesis problem, for such systems are difficult tasks, due to the often large number of agents resulting in a large scale system. In [3] Fax and Murray have used the analogy between the communication topology of a multi-agent system and its graph, and have proved that the stability of a group of $N$ agents is equivalent to the simultaneous stability of $N$ decoupled systems. Their results are used later in [4], [5] and [6] as a basis for different LQR synthesis techniques. However, as such techniques require that the agents exchange also state information; they require observers and large communication overhead and thus are not considered here.

Two recent techniques propose output feedback controller synthesis methods for the controller design problem. The first, proposed in [7], uses a decomposition based technique and results in a synthesis by linear matrix inequalities (LMIs). The second technique [8] treats the communication topology as uncertainty, and using results from robust control theory designs a controller that guarantees the stability of the multi-agent system under any time-varying communication topology and any number of agents. A preliminary versions of the later approach have been presented in [9] and [10].

Although resulting in design problems which can be efficiently solved, these techniques suffer from two problems. The first one is the conservatism incorporated in the controller synthesis for a MAS with known communication topology. In [7] this is due to the need of imposing a common Lyapunov matrix to all LMIs that need to be solved. In [8] the conservatism is due the fact that the approach designs a controller guaranteeing stability under every communication topology.

The second drawback of the above techniques is due to the used decomposition-based technique. To see this, recall that in [3] a transformation matrix $Q$ is used (diagonalizing the Laplacian matrix $L$ describing the communication topology $L = Q\Lambda Q^{-1}$) to diagonalize the closed-loop representation of the MAS. For this purpose either the Schur diagonalization (resulting in an upper triangular system [3]) or the eigenvector diagonalization (resulting in a diagonal system [7]) can be used. However, whereas the diagonalizing transformation is not of importance for stability analysis (analyzing the stability of the $N$ systems on the diagonal) it plays an important role in evaluating the performance. This has been shown in [7], where for the case of an eigenvector diagonalization the following result was derived.

$$\frac{1}{\kappa}\max_i \|\tilde{T}_i(s)\|_\infty \;\leq\; \|\mathbf{T}(s)\|_\infty \;\leq\; \kappa \max_i \|\tilde{T}_i(s)\|_\infty,$$

where $\kappa$ is the condition number of $Q$, $\mathbf{T}(s)$ the performance of the MAS, and $\tilde{T}_i(s)$ are the performances of the $N$ systems from the diagonal of the transformed system. Unfortunately, whereas for the symmetrically decomposable systems considered in [7] $\kappa = 1$, for any other communication topology $\kappa > 1$, and as shown in [10] even for a very simple topology with 3 agents $\kappa = \infty$, and no performance guarantees can be provided. On the other hand, the Schur diagonalization results in an upper triangular system for which no analytic connection between the performance of the MAS and the performance of the decomposed subsystems has been derived, yet.

To avoid the above problems, this paper proposes to directly evaluate the MAS performance during the synthesis procedure, i.e. to minimize $\|\mathbf{T}(s)\|_\infty$. Because the same controller is replicated on each agent, the above synthesis problem is non-convex and cannot be addressed using standard $\mathcal{H}_\infty$ design methods. This paper exploits the replicated structure of the controller and proposes a gradient-based optimization technique. Because the design problem is non-convex and involves computing the $\mathcal{H}_\infty$ norm of a large scale systems (for a large number of agents), a two-step procedure is proposed, where in the first step the robust technique from [8] is used to obtain a stabilizing controller, which is then used to initialize the gradient-based search.

The advantages of the proposed method are twofold - it is capable of obtaining less conservative, and hence better, results than any of the decomposition approaches, and it can serve as a basis for comparing different synthesis methods, as it provides (nearly) optimal results for a given communication topology. A byproduct of the approach worth mentioning is the fact that since the problem is non-convex, one can additionally impose order/structure on the local controller and hence obtain simple control laws (e.g., PID), which are often preferred on agents with limited computation capabilities and power supply, such as mobile robots.

The rest of the paper is organized as follows. In Section II the framework of multi-agent control is reviewed and the distributed control design task is formulated. Section III presents the main contribution of this paper - a gradient-based design for controllers with replicated structure, and further discusses the two-stage procedure for reducing the computational cost of the synthesis. Numerical results with the proposed design approach are given in Section IV. Section V concludes the paper.

The paper uses fairly standard notation: $I_n$ denotes the $n \times n$ identity matrix; $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ are respectively the sets of $n \times m$ real and complex matrices; $A_{ij}$ denotes element $(ij)$ of the matrix $A$; $\otimes$ is the Kronecker product; $\mathcal{F}_L(G(s), K(s))$ denotes the lower linear fractional transformation of $G(s)$ with $K(s)$ ([11]).

## II. Control of Multi-Agent Systems

This section reviews the framework for multi-agent control proposed in [3] and some results from graph-theory [12]. The closed-loop multi-agent system is formed and the distributed controller design problem posed.

### A. Agent and Local Controller

Consider a multi-agent system with $N$ identical LTI agents, each of which is controlled locally as shown in Fig. 1 (i.e., the controller is *distributed* over the MAS). The controller receives both local feedback information $\phi_i$ from the agent (inner-loop), as well as information about the position error $e_i \in \mathbb{R}^p$ of the agent in the formation (outer-loop). The output $y_i \in \mathbb{R}^p$ represents signals sent to or sensed by the other agents.

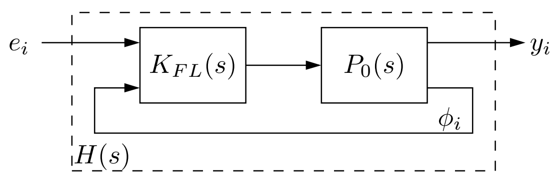


Fig. 1. Local feedback loop of a single agent.

To simplify the derivations and due to space constraints, in the sequel it will be assumed that the local loop is already closed and stabilizes the agent's dynamics (i.e., the system $P(s)$ in Fig. 2 includes the closed inner-loop in Fig. 1 and is stable). However, the controller design techniques presented in the following can also be applied to a simultaneous design for both inner-loop and outer-loop controller. Assume further that the task of the agents is to form a commanded formation, but clearly the presented results are applicable to a general class of problems, where the outputs of the agents do not necessary have spatial interpretation.

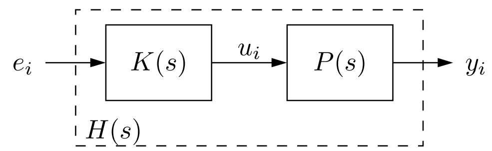


Fig. 2. Single agent and its (outer-loop) controller.

Let the formation-level control error be the normalized sum of the relative errors towards the sensed agents [3]

$$e_i = \frac{1}{|\mathcal{N}_i|} \sum_{j\in\mathcal{N}_i} e_{ij}, \tag{1}$$

where $\mathcal{N}_i$ is the set of agents from which agent $i$ receives information and $|\mathcal{N}_i|$ is the size of this set, i.e. the number of sensed neighbors. The term $e_{ij}$ is the error between the $i$-th and $j$-th agent

$$e_{ij} = (r_i - y_i) - (r_j - y_j) = \bar{r}_{ij} - (y_i - y_j) \tag{2}$$

where $\bar{r}_{ij} \in \mathbb{R}^p$ is the desired distance between the outputs of agents $i$ and $j$. In the case of a leader agent one can define $e_i = r_i - y_i$, thus allowing direct command input to the leader.

### B. Graph Description of the Communication Topology

The communication topology between the agents in the multi-agent system can be described by a directed graph, where each node is an agent, and each directed edge represents an information flow. The normalized Laplacian matrix $L \in \mathbb{R}^{N\times N}$ of such a graph is defined as

$$L_{ij} = \begin{cases} 1, & \text{if } i = j \\ -\frac{1}{|\mathcal{N}_i|}, & j \in \mathcal{N}_i \\ 0, & j \notin \mathcal{N}_i \end{cases} \tag{3}$$

Three important properties of $L$ are

- 0 is always an eigenvalue of $L$,
- The eigenvalues of $L$ lie in an unit disk, centered at $1 + j0$,
- For undirected graphs $L$ has only real eigenvalues.

The results presented in the sequel will be also valid for non-equally weighted communication links, as long as the error signal is normalized.

### C. Closed-Loop Multi-Agent System

One can now build a closed-loop representation of the MAS. From equations (1), (2) and the definition of $L$ it follows that

$$\mathbf{e} = L_{(p)}\left(\mathbf{r} - \mathbf{y}\right) = \bar{\mathbf{r}} - L_{(p)}\mathbf{y} \tag{4}$$

where $\mathbf{e} = \begin{bmatrix} e_1^T & \dots & e_N^T \end{bmatrix}^T$, $\mathbf{r} = \begin{bmatrix} r_1^T & \dots & r_N^T \end{bmatrix}^T$, $\mathbf{y} = \begin{bmatrix} y_1^T & \dots & y_N^T \end{bmatrix}^T$ and $L_{(p)} = L \otimes I_p$, where $\mathbf{e}, \mathbf{y}, \mathbf{r} \in \mathbb{R}^{pN}$. The matrix $L_{(p)}$ is needed so that the correct output channels are compared. Note that since $L$ is singular, $\mathbf{r}$ uniquely determines the relative distances $\bar{\mathbf{r}}$ in the formation, but the converse is not true.

The closed-loop MAS is shown in a compact form in Fig. 3. Note that the only coupling between the systems is via the communication channels, i.e., via $L_{(p)}$.

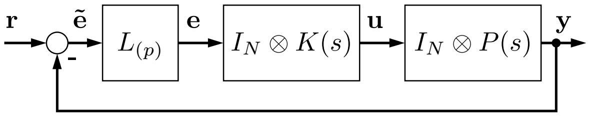


Fig. 3. Closed-loop multi-agent system.

### D. Design Task

In this paper the $\mathcal{H}_\infty$ optimal controller design task is considered, where a mixed-sensitivity approach is used to define the performance requirements [11], [13]. It is important to note that a sensitivity weighting filter $W_S(s)$ can have as input signal either $\tilde{\mathbf{e}}$ or $\mathbf{e}$. Taking the signal before $L_{(p)}$, i.e. $\tilde{\mathbf{e}}$, will weight the absolute error between the agents and their commanded positions. Taking $\mathbf{e}$, on the other hand, will weight the relative error in the desired formation. Because in a MAS without leader the absolute error can be arbitrary large (adding a constant to $\mathbf{r}$ will not change $\mathbf{e}$, but change $\tilde{\mathbf{e}}$), in the sequel weights on $\mathbf{e}$ are imposed. A typical construction of a generalized plant with sensitivity and control-sensitivity weights is shown in Fig. 4, where $\mathbf{w_P} = \mathbf{r}$ are the exogenous inputs and $\mathbf{z_P} = \begin{bmatrix} \mathbf{z_K}^T & \mathbf{z_S}^T \end{bmatrix}^T$ are the performance outputs. Let $\mathbf{K}(s) = I_N \otimes K(s)$ and $\mathbf{T}(s) = \mathcal{F}_L(\mathbf{G}(s), \mathbf{K}(s))$.

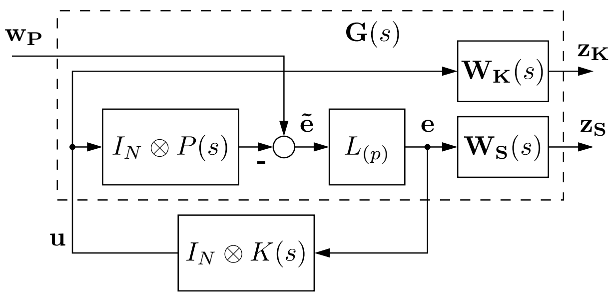


Fig. 4. Interconnection of generalized plant $\mathbf{G}(s)$ of a MAS with sensitivity and control sensitivity filters with a controller $\mathbf{K}(s) = I_N \otimes K(s)$.

Then, for a known and fixed communication topology (i.e., fixed $L$) the design task is

$$\underset{K(s)}{\text{minimize}} \left\| \mathcal{F}_L \left( \mathbf{G}(s), \ I_N \otimes K(s) \right) \right\|_\infty \tag{5}$$

As already discussed in the introduction, solving this problem using a decomposition based approach leads to a conservative design, which cannot guarantee the performance of the MAS. On the other hand, solving (5) is not possible using standard $\mathcal{H}_\infty$ synthesis tools since the replicated structure in $\mathbf{K}(s)$ cannot be imposed. The next section presents the main contribution of this paper - a gradient-based technique exploiting the replicated structure of $K(s)$ in $\mathbf{K}(s)$.

## III. GRADIENT-BASED $\mathcal{H}_\infty$ CONTROLLER DESIGN

Different optimization techniques can be used to attack the design problem in (5), e.g., metaheuristics, Nelder and Mead's method, [14], [15], gradient-based methods [16], [17], etc. In each case, the algorithm will search only over the coefficients of one controller $K(s)$ and at each step will:

1) construct $K(s)$ from the decision variables,
2) construct $\mathbf{K}(s) = I_N \otimes K(s)$,
3) close the loop with $\mathbf{G}(s)$, i.e. obtain $\mathbf{T}(s)$,
4) compute the value of the objective function $\|\mathbf{T}(s)\|_\infty$,
5) update the decision variables and go back to 1).

The obvious advantages of this approach are that it can handle any multi-agent system, work with non-identical agents and facilitate any communication structure. These, however, come at the price of forming large-scale closed-loop systems and computing their $\mathcal{H}_\infty$ norms. This problem will be addressed in Section III-B.

### A. Gradient-Based Design of Controllers with Replicated Structure

Although robust and theoretically able to obtain the global optimum, the direct design approaches (see, e.g., [14]) often require extensive computation and give no guarantee even for local optimality. Two recent methods [16], [17] propose efficient gradient-based approaches for $\mathcal{H}_\infty$ controller synthesis and have proven good results on a set of non-convex low-order and fixed-structure controller synthesis problems [18].

In order to be able to use a gradient-based approach for (5), one should be able to compute the derivative of the objective function with respect to $\mathbf{K}(s)$ and compute the derivative with respect to $K(s)$. The former is already discussed in [16] and [17]. Hence, in the sequel it will be shown how from a given gradient of $\|\mathbf{T}(s)\|_\infty$ with respect to $\mathbf{K}(s)$ one can obtain the gradient of $\|\mathbf{T}(s)\|_\infty$ with respect to $K(s)$ and hence with respect to the decision variables.

First, note that the state-space matrices of $\mathbf{K}(s)$ are obtained from the matrices of $K(s)$ by

$$\begin{aligned} \mathbf{A_K} &= I_N \otimes A_K, & \mathbf{B_K} &= I_N \otimes B_K, \\ \mathbf{C_K} &= I_N \otimes C_K, & \mathbf{D_K} &= I_N \otimes D_K \end{aligned}$$

To be able to compute the gradient of the objective function w.r.t. $K(s)$ one can prove the following Theorem, which is valid not only in the case of $\mathcal{H}_\infty$ norm optimization but for any objective function for which the gradient can be computed.

**Definition 1** *The matrix-gradient of a function* $f(A) : \mathbb{R}^{n\times m} \to \mathbb{R}$ *is*

$$\nabla_A f = \frac{\partial f}{\partial A} = \begin{bmatrix} \frac{\partial f}{\partial A_{11}} & \cdots & \frac{\partial f}{\partial A_{1m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial A_{n1}} & \cdots & \frac{\partial f}{\partial A_{nm}} \end{bmatrix}.$$

**Theorem 1** *Given functions* $A(x) : \mathbb{R} \to \mathbb{R}^{n\times m}$*;* $A(x) = A_0 + P \otimes x$ *and* $f(A) : \mathbb{R}^{n\times m} \to \mathbb{R}$*, where* $x \in \mathbb{R}$*;* $A_0, P \in \mathbb{R}^{n\times m}$ *and* $P$ *is a pattern matrix. Let the matrix-gradient* $\nabla_A f$ *of* $f$ *with respect to the elements of* $A$ *be known. Then the gradient of* $f$ *with respect to* $x$ *is*

$$\nabla_x f = \sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}\frac{\partial f}{\partial A_{ij}} \tag{6}$$

The proof is given in Appendix.

The above theorem states that the gradient w.r.t. $x$ is the sum of the elements of the matrix gradient w.r.t. $A$, weighted by the corresponding elements in the pattern matrix. For example, the gradient of the closed-loop $\mathcal{H}_\infty$ norm $f = \|\mathbf{T}(s)\|_\infty$ w.r.t. element $hk$ of the output matrix $C_K \in \mathbb{R}^{p\times n}$ is

$$\nabla_{C_{K_{hk}}} f = \sum_{i=1}^{pN}\sum_{j=1}^{nN} P_{ij}\frac{\partial f}{\partial \mathbf{C}_{\mathbf{K}_{ij}}} = \sum_{i=0}^{N-1}\frac{\partial f}{\partial \mathbf{C}_{\mathbf{K}_{(ip+h)(in+k)}}},$$

where $P = I_N \otimes (e_h e_k^T)$, $e_h$ denote column $h$ of $I_p$, and $e_k$ column $k$ of $I_n$.

### B. Two-stage Design Method

As discussed earlier, the main drawback of the direct search over $K(s)$ and the evaluation of the performance of the whole MAS is the evaluation of the $\mathcal{H}_\infty$ norm of a large-scale system, which is computationally expensive for systems with a large number of agents. Although the cost of a single $\mathcal{H}_\infty$ norm evaluation cannot be reduced without either increasing the conservativeness of the design or/and loosing the guarantees of the closed-loop performance, one can significantly reduce the required number of closed-loop $\mathcal{H}_\infty$ norm evaluations by using an appropriate initialization procedure.

Although both the algorithms from [7] and [8] can be used for obtaining an initial controller, the latter one is applied here, because:

- In the general case (directed communication) the approach from [7] has to solve $N$ LMIs simultaneously vs. one LMI for the method in [8]. Indeed for undirected communication the method in [7] reduces to only 2 LMIs, but this is restrictive since it does not allow the presence of leader agents in the MAS.
- The approach in [8] obtains a controller stabilizing the MAS under any communication topology and hence the initialization step needs not be repeated when the communication topology is changed and a new controller needs to be synthesized.

In the following we briefly recall the MAS stability criterion proposed in [3] and the distributed controller synthesis method from [8].

### C. Stability of a Multi-Agent System

Let the dynamics $P(s)$ of a single agent be given by a state-space model

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i \qquad i = 1, \ldots, N. \end{aligned} \tag{7}$$

The following results are adapted from [3].

**Definition 2** *A MAS is called stable, if all eigenvalues of the closed-loop system (Fig. 3) are in the left half-plane.*

**Theorem 2** *A controller* $K(s)$ *stabilizes the closed-loop multi-agent system Fig. 3 if and only if it simultaneously stabilizes the set of* $N$ *systems*

$$\begin{aligned} \dot{\tilde{x}}_i &= A\tilde{x}_i + B\tilde{u}_i \\ \tilde{y}_i &= \lambda_i C\tilde{x}_i \qquad i = 1, \ldots, N \end{aligned} \tag{8}$$

*where* $\tilde{e}_i = -\tilde{y}_i$*,* $\tilde{u}_i = K(s)\tilde{e}_i$ *and* $\lambda_i$ *are the eigenvalues of the normalized Laplacain matrix* $L$*.*

### D. Robust Control Approach

In [8] the above theorem is used together with the fact, that all eigenvalues of $L$ are in a unit disk centered at $1 + j0$ to prove the following Theorem.

**Theorem 3** *A controller* $K(s)$ *stabilizes a multi-agent system for any number of agents and under any (arbitrary fast) time-varying communication topology if there exists an invertible matrix* $D \in \mathbb{R}^{p\times p}$*, such that*

$$\|DT(s)D^{-1}\|_\infty < 1,$$

*where* $T(s) = (I_p + H(s))^{-1}H(s)$*, and* $H(s) = P(s)K(s)$ *is as shown in Fig. 2.*

By constructing the appropriate generalized plant and using standard robust control tools [11], [13], this theorem can be directly used for controller synthesis.

### E. Robust Performance Controller Design

As discussed in Section II-D, performance requirements on the MAS can be expressed via the mixed-sensitivity framework. Such requirements can be added also to the robust approach in Theorem 3 by representing $\lambda_i = 1 + \delta_i$ (where $|\delta_i| \le 1$) and using that after the closed-loop diagonalization, $\mathbf{e} = L_{(p)}(\mathbf{r} - \mathbf{y})$ becomes $\tilde{e}_i = \lambda_i(\tilde{r}_i - \tilde{y}_i) = (\tilde{r}_i - \tilde{y}_i) + \delta_i(\tilde{r}_i - \tilde{y}_i)$. Then by dropping the index $i$, since it should hold for all $i = 1, \ldots, N$, one can construct the following generalized plant and augment it by shaping filters:

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ z &= -C\tilde{x} + I_p\tilde{r} \\ \tilde{e} &= -C\tilde{x} + I_p\tilde{r} + I_p w \end{aligned} \tag{9}$$

where $w = \delta\, I_p\, z$. For more details see [10].

Although, in accordance with the discussion in Section I, the performance measure obtained using this synthesis technique will not be the worst case performance of the MAS it nevertheless leads to a good initial controller for the gradient-based algorithm.

## IV. Numerical Results

In this section the above design method is applied to the controller synthesis problem for a satellite formation. The gradient-based algorithm is based on HIFOO [16] due to its good performance on a set of benchmark problems [18] and the available Matlab implementation. The handling of the repeated controller structure in $\mathbf{K}(s)$ is as discussed in Theorem 1.

Next, the satellite reconfiguration problem discussed in [7], [19], [20] is considered, where each agent is a satellite, and the dynamic equations describe the satellite motion when perturbed from a circular orbit. The 3-dimensional dynamics of the satellites can be found in [19] and [20]. Because the z-axis is decoupled from the others and not of interest in the control problem, it is assumed that it is controlled independently. Hence the focus is on the satellite motion in the x-y plane. The local feedback $\phi_i$ (see Fig. 1) is considered to contain the system states, whereas the formation-level feedback $y_i$ contains (only) the satellite coordinates. An LQR controller is designed for the inner-loop, yielding a $2 \times 2$ system $P(s)$ with 4 states. The outer-loop controller is designed using mixed-sensitivity $\mathcal{H}_\infty$ synthesis with sensitivity and control-sensitivity, as shown in Fig. 4.

For the purpose of comparison, the design is carried out for an undirected chain interconnection of $N$ agents, i.e. $\mathcal{N}_i = \{i-1, i+1\}$, $i = 2, \ldots, N-1$, $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_N = \{N-1\}$ with $N = \{2, 4, 6, 8\}$ agents. In all cases 0 and 2 are eigenvalues of $L$, which justifies solving just 2 LMIs with the decomposition-based approach of [7]. The results for the closed-loop $\mathcal{H}_\infty$ norm are visualized in Fig. 5. Besides the three methods discussed earlier, the figure also shows the results from a centralized controller, which constitutes the achievable best result. From the figure one can see that the results of the gradient-based approach improves over the robust one and is close to the results from the decomposition-based technique. In fact it achieves slightly better results than the latter one for $N = \{2, 4, 8\}$, and slightly worse ones for $N = 6$. This is due to the fact that the gradient-based technique attacks a non-convex problem and no guarantee of convergence to the global optimum can be given. Better performance of the gradient-based technique can be expected on multi-agent systems with directed communication topologies, where the conservativeness due to the common Lyapunov matrices will play a stronger role. As for the computation time: whereas by the robust and decomposition-based methods (for the undirected communication topologies) the synthesis is in the range of seconds for the gradient-based it increases with the number of agents (due to the computation of the $\mathcal{H}_\infty$ norm of ever larger-scale systems) and for $N = 8$ is 380 s. This suggests that the technique is mainly applicable to MAS with relatively small (less than 20) agents.

Finally the robust-based and the gradient-based designs are applied to the problem of fixed-order controller synthesis. Second order controllers are designed for a MAS with an undirected cyclic communication topology, i.e., $\mathcal{N}_i = \{i-1, i+1\}$, $i = 2, \ldots, N-1$, $\mathcal{N}_1 = \{N, 2\}$, $\mathcal{N}_N = \{N-1, 1\}$. Note that as the design results in a non-convex optimization problem, the decomposition-based approach is not applicable. The closed-loop $\mathcal{H}_\infty$ norms with the obtained controllers are presented in Fig. 6. One can clearly see that by restricting the controller order, the performance achievable by the robust-control approach degrades, but the gradient-based technique is capable to improve it and in fact achieves better performance than the full-order robust controller.

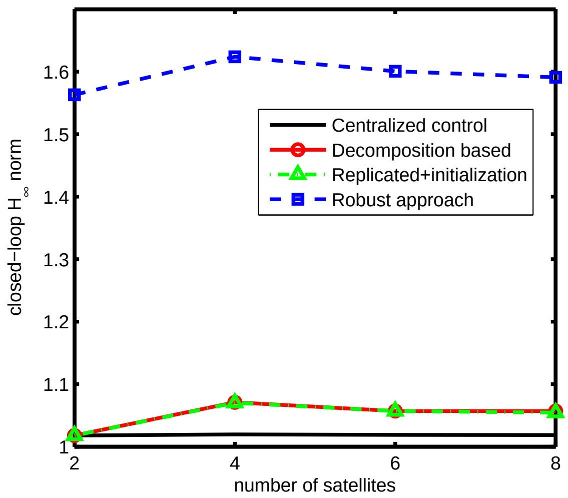


Fig. 5. Closed-loop $\mathcal{H}_\infty$ norm for design for MAS with undirected chain communication topology.

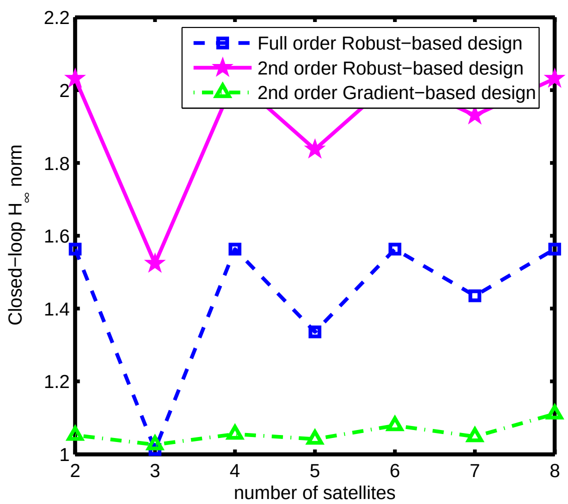


Fig. 6. Results from low-order controller synthesis.

## V. Conclusions

In this paper a gradient-based technique for $\mathcal{H}_\infty$ distributed controller synthesis for a multi-agent system with known communication topology has been presented. The technique exploits the replicated controller structure and directly optimizes over the controller coefficients by evaluating the closed-loop $\mathcal{H}_\infty$ norm. This allows the method to provide a guarantee of closed-loop performance while at the same time avoiding conservatism in the design due to a common Lyapunov matrix. However, as the resulting design problem is a non-convex one and the synthesis for MAS with many agents leads to evaluations of the $\mathcal{H}_\infty$ norm of large scale systems, the approach is mainly applicable to smaller-scale systems.

Two byproducts of the proposed technique worth mentioning are that it can handle restrictions on the order and structure of the distributed controller, and that it is applicable to a general class of $\mathcal{H}_\infty$ controller synthesis problems, where the controller has a replicated structure. Examples of the latter are control of redundant systems (e.g., actuators in an aircraft) or control of symmetric systems (e.g., breaks control in a car).

## APPENDIX
## PROOF OF THEOREM 1

Let $e_i$ denote column $i$ of $I_n$, and $e_j$ column $j$ of $I_m$. Note that

$$P \otimes x = \sum_{i=1}^{n} \sum_{j=1}^{m} e_i \left(P_{ij} x\right) e_j^T$$

Applying the chain rule (see, e.g., [21]) for a single element $B(x) = e_i \ (P_{ij}\, x) \ e_j^T$, of the above double sum:

$$\begin{aligned}
\nabla_x f &= \langle \nabla_B f \, , \nabla_x B \rangle \\
&= \text{tr}\left( \left(\nabla_B f\right)^T \ e_i \, P_{ij} \, e_j^T \right) = \text{tr}\left( P_{ij} \ e_j^T \ \left(\nabla_B f\right)^T \ e_i \right) \\
&= e_j^T \ P_{ij} \ \left(\nabla_B f\right)^T \ e_i
\end{aligned}$$

where $\langle X, Y \rangle = \text{tr}\left(X^T Y\right)$ is the inner product of $X$ and $Y$, and "tr" denotes the trace (recall that $\text{tr}(XY) = \text{tr}(YX)$). Then, rewriting

$$\nabla_A f = \sum_{i=1}^{n} \sum_{j=1}^{m} e_i \, \frac{\partial f}{\partial A_{ij}} \, e_j^T$$

and using that $e_j^T \, e_j = 1$, the matrix gradient with respect to $x$ is

$$\begin{aligned}
\nabla_x f &= \langle \nabla_A f \, , \nabla_x A \rangle \\
&= \text{tr}\left( \sum_{i=1}^{n} \sum_{j=1}^{m} e_j \left( \frac{\partial f}{\partial A_{ij}} \right)^T \ e_i^T \quad \sum_{i=1}^{n} \sum_{j=1}^{m} e_i P_{ij} e_j^T \right) \\
&= \text{tr}\left( \sum_{i=1}^{n} \sum_{j=1}^{m} e_j^T e_j \ P_{ij} \frac{\partial f}{\partial A_{ij}} \, e_i^T e_i \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij} \frac{\partial f}{\partial A_{ij}}
\end{aligned}$$

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