# Robust Stability of a Multi-Agent System under Arbitrary and Time-Varying Communication Topologies and Communication Delays

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***Abstract*— This paper considers formation control of a group of identical agents that can communicate with each other. A necessary and sufficient condition for robust stability in the case of arbitrary time-invariant communication topologies and sufficient conditions in the case of arbitrary time-varying topology and communication delays are derived, that reduce the stability analysis and controller design problems to analysis or synthesis, respectively, for a single agent.**

***Index Terms*— Multi-Agent System, Formation Control, Robust Control, Time-Varying Topology, Communication Delay.**

## I. Introduction

This paper considers formation control of multi-agent systems (MAS). MAS are dynamical systems comprising two or more physically independent agents (e.g., mobile robots, etc.) that can communicate and interact with each other (see, e.g., [1]).

The problem of cooperative control of such MAS is studied by Fax and Murray in [2], where using a decomposition approach it is shown that the stability of a MAS with $N$ agents and a known and fixed communication topology is equivalent to the simultaneous stability of $N$ transfer functions, each representing a single agent scaled by a (possibly complex) eigenvalue $\lambda_i$ of the Laplacian matrix $\mathcal{L}$ used to describe the communication topology. Their results are used in [3], [4], [5] where distributed Linear Quadratic Regulator (LQR) designs are proposed, and in [6] and [7] where output feedback control laws based, respectively, on a decomposition principle and a robust control approach are presented.

In this paper we consider the communication topology and communication delays as unknown and apply robust control techniques, using the fact that for an arbitrary topology the eigenvalues of $\mathcal{L}$ are contained in a unit disk centered at 1 (the Perron disk). In this way we address the fact that the communication topology between mobile agents can depend on communication range limitations, obstacles in the environment, communication disturbances or/and time-varying communication delays.

In [7] sufficient condition for the stability of a MAS under an uncertain communication topology is derived, that reduces the stability analysis to an $\mathcal{H}_\infty$ condition. However, as pointed out in [7], in the case of time-varying communication topology the decomposition-based method of [2] cannot be applied because then the required decomposing similarity transformation $Q$ is also time-varying and since, in general, $Q(t)Q^{-1}(t+1) \neq I$.

In order to address these problems, in this work we take a different approach by avoiding the system decomposition. In this way a condition that is not only sufficient but also necessary for stability of a MAS with an arbitrary time-invariant topology and communication delays is obtained. For MAS with time-varying topology we derive a sufficient condition for stability expressed by a scaled system $l_1$ norm and show that for a certain set of topologies an alternative condition for stability is the scaled system $\mathcal{H}_\infty$ norm. Additionally, the obtained stability analysis conditions are independent of the number of agents and reduce to stability analysis of a single agent with uncertainty. Furthermore, the proposed method reduces the controller synthesis problem to synthesis for a single agent.

It is important to note that the stability problem studied here can be seen as a prerequisite for studying consensus and synchronization, see, e.g., [8]–[9], since even if agents become disconnected the stability property will guarantee that no unstable behavior will occur in the system. This allows to separate the requirements on the topology, necessary for consensus and synchronization, from stability analysis under arbitrary fixed and time-varying topologies. However, if additional requirements on the topology are imposed (e.g., strong connectedness as in [7], or uniform connectedness as in [10]) and when the agents have stable dynamics, statements about consensus/synchronization can be derived from the stability results presented here. At the same time, performing a synthesis using the proposed methodology is attractive not only because it requires controller design for a single agent, but also because even for moderate size MAS the eigenvalues associated with an uncertain topology and with uncertain communication delays $\mathcal{L}$ rapidly fill the Perron disk, and because the obtained controllers are well suited for initializing controller synthesis techniques tailored to a specific topology/set of topologies (see, e.g., [11]).

In the light of the above, the presented results differ from previous work in several aspects. Firstly, whereas in [2]–[7], [10] communication delays are not considered, or are required to be fixed, known and accounted for in the plant model, the presented approach considers arbitrary, unbounded and time-varying communication delays. The presented results differ from the ones in [9], [12] since agents with general LTI dynamics, rather than single integrators are considered; from the results in [10], [13]–[14] since no

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restrictions are imposed on the communication topology; as well as from the results in [10], [15], [16] in that the proposed techniques provide both systematic analysis and synthesis methodologies. Additionally, in contrast to the techniques in [10], [15], which require that estimated agent state-vectors are communicated, the proposed method uses only the output signals of the agents. Finally, whereas the approaches in [3], [4], [17] consider undirected communication topologies, or require special attention for directed ones, as in [6], the method proposed here is designed for directed communication topologies and includes the undirected ones as a special case.

Finally, an information flow approach presented in [2] provides a method for stabilizing MAS with arbitrary constant communication topologies, as long as a specific *information-flow* structure is used and the agents communicate over a network. The proposed method differs from this in three aspects: it allows stability analysis of MAS both with and without such an information-flow structure; it allows agents to obtain/exchange information by means of sensor data (video camera, proximity sensors, etc.), and, most importantly, allows for time-varying topologies and communication delays.

The paper is structured as follows. In Section II the multi-agent system framework used here is briefly reviewed. Section III presents the main results of this paper - a necessary and sufficient condition for stabilizing a MAS with or without communication-delays, under any fixed topology and communication delays, as well as sufficient conditions for stability under time-varying topology and delays. The design method is illustrated by an example in Section IV. Section V concludes the paper.

The following notation will be used: $\mathbb{R}^{p\times q}$, $\mathbb{C}^{p\times q}$ are the sets of $p\times q$ real and complex matrices, $\mathbb{Z}$ is the set of all integers, and $\mathbb{N}_0$ is the set of non-negative integers; $I_p$ denotes the $p\times p$ identity matrix, 0 a zero matrix with appropriate dimensions, and j is the complex unit. For a matrix $A\in\mathbb{C}^{n\times n}$ the spectrum is denoted by $\operatorname{spec}(A)$, the spectral radius by $\rho(A)$ and the element in row $i$ and column $k$ by $A_{ik}$. The discrete-time variable is $t\in\mathbb{Z}$ and all signals are zero for $t<0$. The lower linear fractional transformation of two transfer functions $P(z)$ and $K(z)$ with appropriate number of inputs and outputs is denoted by $\mathcal{F}_L(P(z),K(z))$, where $z$ is a complex variable. The Kronecker product is denoted by $\otimes$. Additionally the following sets are defined $\mathsf{D}:=\{\chi\mid\chi\in\mathbb{C},|\chi|<1\}$ (open unit disk); $\mathsf{C}:=\{\chi\mid\chi\in\mathbb{C},|\chi|=1\}$ (unit circle); $\bar{\mathsf{D}}=\mathsf{D}\cup\mathsf{C}$ (closed unit disk) and $\mathsf{B}:=\begin{bmatrix}-1, & 1\end{bmatrix}$. Furthermore, $\ell^\infty$ denotes the space of all bounded sequences of real numbers; $\|x\|_\infty=\sup_t|x(t)|$ denotes the infinity norm of a signal $x(t)\in\ell^\infty$; $\ell^\infty_p$ denotes the space of $p$-tuples of elements of $\ell^\infty$, i.e., $p$-dimensional signals. Finally, we recall that for a stable LTI system $H(z)$ with $p$ inputs and $q$ outputs and impulse response matrix $h(t)$ the $l_1$ norm is defined as $\|H(z)\|_{l_1}=\max_j\sum_{i=1}^p\sum_{t=0}^\infty|h_{ji}(t)|$ and is equal to the norm induced by the signal infinity norm. The $\mathcal{H}_\infty$ norm of $G(z)$ is denoted by $\|G(z)\|_{\mathcal{H}_\infty}$.

## II. Multi-Agent System

In this section we recall results from graph-theory and a multi-agent framework [2] that will be used to derive our main result in the next section.

### A. Local Dynamics and Stability

Consider a group of $N$ identical agents with a causal discrete-time LTI model $P(z)$ and a local LTI controller $K(z)$, as shown in Fig. 1, where $u_i\in\mathbb{R}^h$ is the control

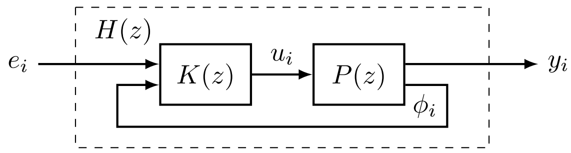


Fig. 1. Local feedback of single agent and its controller

signal for agent $i$; $\phi_i\in\mathbb{R}^m$ is the locally available measured output; $y_i\in\mathbb{R}^p$ is the output that is communicated to or sensed by other agents; $e_i\in\mathbb{R}^p$ is the formation control error, which will be defined later. For example, in a mobile robot $\phi_i$ might be joint positions, velocities, etc., $y_i$ the position of the robot in the operating space, and $u_i$ voltages applied to the driving motors. Hence, the local controller of each agent uses both local signals $\phi_i$ (for low-level control and autonomous stability) and multi-agent error signal $e_i$ (expressing, e.g., deviation of the agent from its commanded position within a formation). The transfer function from $e_i$ to $y_i$ is $H(z)=\mathcal{F}_L\left(P(z)K(z),\,I_m\right)$.

### B. Communication Topology and Control Error

Assume, initially, that there are no communication delays. The control error is defined [2] as the average of the errors between the outputs of agent $i$ and the outputs of agents $k$:

$$e_i=\frac{1}{|\mathsf{N}_i|}\sum_{k\in\mathsf{N}_i}e_{ik}, \tag{1}$$

where $k\in\mathsf{N}_i$, $\mathsf{N}_i$ is the set of agents from which agent $i$ receives information and $|\mathsf{N}_i|$ its cardinality. The term $e_{ik}$ is the error between the $i$-th and $k$-th agent

$$e_{ik}=\bar{r}_{ik}-(y_i-y_k), \tag{2}$$

where $\bar{r}_{ik}\in\mathbb{R}^p$ is a prescribed difference between the outputs. In a formation control problem $y_i$ may be the position of agent $i$ and $\bar{r}_{ik}$ the desired distances to agent $k$.

The communication topology within the MAS can be represented as a directed-graph, where the nodes are the agents and the edges are the communication links. The normalized Laplacian matrix $\mathcal{L}$ of the graph describes the graph (and in turn the communication-topology) and is defined as

$$\mathcal{L}_{ik}=\begin{cases}1, & \text{if } i=k \text{ and } |\mathsf{N}_i|\neq 0\\ -\frac{1}{|\mathsf{N}_i|}, & \text{if } k\in\mathsf{N}_i,\\ 0, & \text{otherwise,}\end{cases} \tag{3}$$

where $N$ is the number of agents and $\mathsf{N}_i$ is the set of agents from which agent $i$ receives information. The results presented below hold also if different weights are assigned to the communication channels, as long as the error signals are normalized.

Let $\mathcal{L}_{(p)} = \mathcal{L} \otimes I_p$, $\boldsymbol{e} = \begin{bmatrix} e_1^\mathsf{T} & \dots & e_N^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $\boldsymbol{y} = \begin{bmatrix} y_1^\mathsf{T} & \dots & y_N^\mathsf{T} \end{bmatrix}^\mathsf{T}$. Let $\boldsymbol{r}$ denote a reference signal, that directly defines a commanded value for the outputs of the agents and let $\bar{\boldsymbol{r}}$ denote a reference for the relative difference between the outputs of the agents. Then using (1) and (3) one can write

$$\boldsymbol{e} = \bar{\boldsymbol{r}} - \mathcal{L}_{(p)}\boldsymbol{y} = \mathcal{L}_{(p)}(\boldsymbol{r} - \boldsymbol{y}) = \mathcal{L}_{(p)}\boldsymbol{\eta},$$

where $\boldsymbol{\eta} = \boldsymbol{r} - \boldsymbol{y}$ is an absolute position error. Since $\mathcal{L}$ is not invertible, to the same relative reference $\bar{\boldsymbol{r}}$ correspond infinitely many absolute references $\boldsymbol{r}$. The closed-loop interconnection of the MAS is shown in Fig. 2. In the following a MAS will be called stable if the corresponding closed-loop representation (Fig. 2) has all its poles inside the unit disk $\mathsf{D}$. The transfer function from $\boldsymbol{r}$ to $\boldsymbol{y}$ is $\left(I_{pN} + \boldsymbol{H}(z)\mathcal{L}_{(p)}\right)^{-1}\boldsymbol{H}(z)\mathcal{L}_{(p)}$, where $\boldsymbol{H}(z)$ is a block-diagonal transfer function with $H(z)$ repeated $N$ times along the diagonal, i.e., $\boldsymbol{H}(z) = I_N \otimes H(z)$.

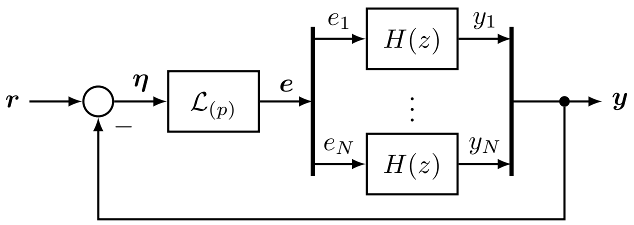


Fig. 2. Closed-loop representation of a multi-agent system.

*C. Communication Time Delays*

Assume now that the communication channel from agent $k$ to agent $i$ is associated with a time-delay $\tau_{ik} \in \mathbb{N}_0$, for each $i, k = 1, \dots, N$, $i \neq k$, but the internal signal $\phi_i$ is undelayed (i.e., a MAS without self-delay [15]). Then one can define a frequency dependent normalized graph-Laplacian as

$$\mathcal{L}_{ik}(z) = \begin{cases} 1, & \text{if } i = k \text{ and } |\mathsf{N}_i| \neq 0 \\ -\frac{1}{|\mathsf{N}_i|} z^{-\tau_{ik}}, & \text{if } k \in \mathsf{N}_i, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

In this case (2) becomes $e_{ik}(t) = \bar{r}_{ik}(t) - (y_i(t) - y_k(t - \tau_{ik}))$.

Define the frequency-dependent normalized adjacency matrix of a graph as $\mathcal{A}(z) = \mathcal{L}(z) - I_N$ and let $\delta_i(\omega) \in \text{spec}\left(\mathcal{A}(e^{\mathrm{j}\omega})\right)$. We then have the following result which will be used in Section III.

**Lemma 1** *For each frequency $\omega \in \mathbb{R}$ the eigenvalues $\delta_i$ of the frequency dependent normalized adjacency matrix $\mathcal{A}(e^{\mathrm{j}\omega})$ satisfy $\delta_i(\omega) \in \bar{\mathsf{D}}$.*

*Proof:* By Gershgorin's disk theorem [18] for a fixed $\omega$ the eigenvalues $\delta_i(\omega)$, $i = 1, \dots, N$ of $\mathcal{A}(e^{\mathrm{j}\omega})$ satisfy

$$|\delta_i(\omega) - \mathcal{A}_{ii}(e^{\mathrm{j}\omega})| \leq \sum_{k=1}^{N} |\mathcal{A}_{ik}(e^{\mathrm{j}\omega})|.$$

Using that $\mathcal{A}_{ii}(e^{\mathrm{j}\omega}) = \mathcal{L}_{ii}(e^{\mathrm{j}\omega}) - 1 = 0$ and that for each $\tau_{ik} \in \mathbb{N}_0$ and $\forall\omega$

$$|\mathcal{A}_{ik}(e^{\mathrm{j}\omega})| = \left| -\frac{1}{|\mathsf{N}_i|} e^{-\mathrm{j}\omega\tau_{ik}} \right| = \frac{1}{|\mathsf{N}_i|} |e^{-\mathrm{j}\omega\tau_{ik}}| = \frac{1}{|\mathsf{N}_i|},$$

leads to $|\delta_i(\omega)| \leq 1$. ■

## III. Stability under Fixed and Time-Varying Communication Topology

In [2] a decomposition-based stability condition is used to derive a Nyquist-like stability criterion for SISO agents and a spectral-radius condition for MIMO agents, subject to a known and fixed communication topology without communication delays. Here we take up the MIMO case and consider not a particular communication topology, but all possible ones, and allow arbitrary and different delays $\tau_{ik} \in \mathbb{N}_0$.

In the following the dependence on $t$, $z$ or $\omega$ will be dropped when clear from the context.

### *A. Stability under any Fixed Topology and Communication Delays*

By substituting $\mathcal{L}(z) = I_N + \mathcal{A}(z)$ in the closed-loop equations of the MAS, one can show that the MAS is stable if and only if the transfer functions $H(z)$, $T(z) = \left(I_p + H(z)\right)^{-1} H(z)$ and $\left(I_{pN} + \boldsymbol{T}(z)\mathcal{A}_{(p)}(z)\right)^{-1} \boldsymbol{T}(z)\mathcal{A}_{(p)}(z)$ are stable, where $\boldsymbol{T}(z) = I_N \otimes T(z)$ and $\mathcal{A}_{(p)} = \mathcal{A} \otimes I_p$. Alternatively, one can write the closed-loop equations as $\left(I_{pN} + \boldsymbol{T}_{\mathcal{A}}(z)\right)^{-1} \boldsymbol{T}_{\mathcal{A}}(z)$ (see Fig. 3), where $\boldsymbol{T}_{\mathcal{A}}(z) = \mathcal{A}(z) \otimes T(z)$ has a block form, with block $ik$ being

$$\boldsymbol{T}_{\mathcal{A},ik}(z) = \begin{cases} -\frac{z^{-\tau_{ik}}}{|\mathsf{N}_i|} T(z), & \text{if } k \in \mathsf{N}_i, \\ 0, & \text{otherwise.} \end{cases}$$

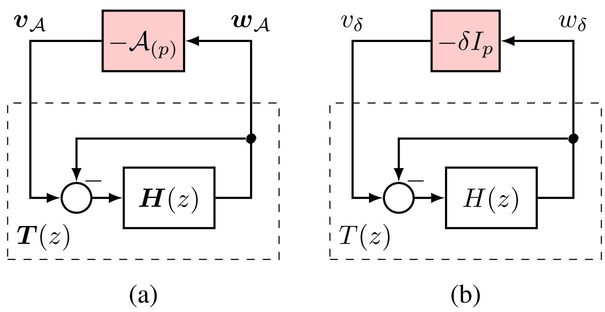


Fig. 3. Interconnection of $\boldsymbol{T}(z)$ and $\mathcal{A}_{(p)}(z)$.

The following lemma will be used later on.

**Lemma 2** *Let $X \in \mathbb{C}^{N \times N}$, and $Y \in \mathbb{C}^{p \times p}$. Then*

$$\mathit{spec}\,(X \otimes Y) = \left\{ \lambda_i \theta_k \;\middle|\; \begin{array}{l} \lambda_i \in \mathit{spec}\,(X)\,,\ i \in [1, N], \\ \theta_k \in \mathit{spec}\,(Y)\,,\ k \in [1, p] \end{array} \right\}.$$

The proof can be found in [19]. Now we can show the following.

**Theorem 1** *The following statements are equivalent:*

(a) *The multi-agent system in Fig. 2 is stable under any fixed topology, for any fixed communication time-delays $\tau_{ik} \in \mathbb{N}_0$ and for any number of agents;*

(b) *The structured singular value of $T(z)$ satisfies $\mu_{\Delta}(T(z)) < 1$, for all $z$ on the unit circle, where $\Delta := \{\delta I_p \mid \delta \in \bar{\mathsf{D}}\}$.*

*Proof:* First we prove (b)⇒(a). Let $\mu_{\Delta}(T(z)) < 1$, $\forall z \in \mathsf{C}$ hold. Then, from the definition of the structured singular value (cf. [20]) and the fact that the uncertain block $\Delta$ has a diagonal structure it follows that $\sup_{z\in\mathsf{C}} \mu_{\Delta}(T(z)) = \sup_{\omega\in[-\pi,\pi]} \rho\left(T(e^{j\omega})\right) < 1$. Since, according to Lemma 1, spec $\left(\mathcal{A}(e^{j\omega})\right) \in \bar{\mathsf{D}}$, $\forall\omega$, by applying Lemma 2 to $\boldsymbol{T}_{\mathcal{A}}(e^{j\omega}) = \mathcal{A}(e^{j\omega}) \otimes T(z)$ for any $\omega \in \mathbb{R}$ it follows that $\rho\left(\boldsymbol{T}_{\mathcal{A}}(e^{j\omega})\right) < 1$, $\forall\omega$ and hence by the spectral radius theorem (cf. [20]) it follows that the closed-loop MAS is stable.

Next we prove (a)⇒(b). Let the MAS be stable under all communication topologies with any communication delays. Then according to the discrete-time multi-variable Nyquist stability test [21, Chap. 11] it follows that the Nyquist plot of $\det(I_{pN} + \boldsymbol{T}_{\mathcal{A}}(z))$

- does not pass through the origin, $\forall\omega$,
- does $q$ anti-clockwise encirclements of the origin, where $q$ denotes the number of unstable poles of $\boldsymbol{T}_{\mathcal{A}}(z)$.

The first of the above conditions requires that

$$(1+\delta\theta) \neq 0,\ \forall\delta \in \text{spec}\left(\mathcal{A}(e^{j\omega})\right),\ \forall\theta \in \text{spec}\left(T(e^{j\omega})\right),\ \forall\omega. \quad (5)$$

Since the MAS is stabile under any communication topology and delays, it is in particular stable with the following topologies without communication delays:

1) MAS with one agent. Since there is no communication it follows that $H(z)$ is stable.
2) MAS with $N = 2$ agents, and one communication link from agent 1 to agent 2. In that case $\boldsymbol{T}_{\mathcal{A}}(z) = \begin{bmatrix} 0 & 0 \\ -T(z) & 0 \end{bmatrix}$ and hence $T(z)$ is a stable transfer function. Then by applying Lemmas 1 and 2 to $\boldsymbol{T}_{\mathcal{A}}(z)$ it follows that $\boldsymbol{T}_{\mathcal{A}}(z)$ does not have unstable poles for any topology and the second of the above Nyquist conditions simplifies to "does not encircle the origin".
3) A MAS with a directed, cyclic communication topology with $N$ agents, where $\mathsf{N}_i = \{i+1\}$, $\forall i = 1, \ldots, N-1$ and $\mathsf{N}_N = \{1\}$. The eigenvalues of the Laplacian are $\lambda_i = 1 - e^{j\frac{2\pi i}{N}}$ (cf. [2]) and thus $\delta_i = -e^{j\frac{2\pi i}{N}}$, $\forall i = 1, \ldots, N$, are equidistant points on the unit circle, $|\delta_i| = 1$, $\forall i$. Let the number of agents increase as $N \to \infty$. Then the stability under this topology is equivalent to (5) being satisfied for all $\delta \in \mathsf{C}$.
4) A MAS with a cyclic undirected communication topology. The eigenvalues of $\mathcal{L}$ are $\lambda_i = 1 - \cos(\frac{2\pi i}{N})$, $\forall i = 1, \ldots, N$. Let the number of agents increase as $N \to \infty$. Then the stability under this topology is equivalent to (5) for all $\delta \in \mathsf{B}$.

The important features of the last three topologies are that $T(z)$ is a stable transfer function, hence analytic on $\mathsf{C}$, $T(z)$ does not have complex eigenvalues with magnitude 1, as well as real eigenvalues with magnitude greater than 1, $\forall z = e^{j\omega}$, $\forall\omega$, i.e., all $\theta \in \text{spec}\left(T(e^{j\omega})\right)$ satisfy

$$\theta \neq 1/\delta, \quad \forall\omega, \forall\delta \in \{\mathsf{C} \cup \mathsf{B} \backslash 0\}, \quad (6)$$

where the case $\delta = 0$ is covered by the stability of $H(z)$ in the first of the above topologies.

Additionally, note that in the case of topologies with communication delays (5) states that $T(e^{j\omega})$ does not have a complex eigenvalue at $\theta(e^{j\omega}) = 1/\delta(e^{j\omega}) \in \text{spec}\left(\mathcal{A}(e^{j\omega})\right)$, $\forall\omega$. Since $|\delta(e^{j\omega})| \leq 1$, $\forall\omega$ it follows that all those eigenvalues are with magnitudes greater one.

Assume now that there exist frequencies $\omega_a$ and $\omega_c$, $0 \leq \omega_a < \omega_c \leq \pi$, s.t. $\rho\left(T(e^{j\omega_a})\right) > 1$ and $\rho\left(T(e^{j\omega_c})\right) < 1$ (or vice versa). Then, since $T(e^{j\omega})$ is analytic on the unit disk $z = e^{j\omega}$, $\forall\omega$ there must exist a frequency $\omega_b$, $\omega_a < \omega_b < \omega_c$, s.t. $\rho\left(T(e^{j\omega_b})\right) = 1$ and hence there must exist an eigenvalue $\theta_k \in \text{spec}\left(T(e^{j\omega_b})\right)$, s.t. $|\theta_k| = 1$. This however contradicts the requirement that there is no eigenvalue of $T(e^{j\omega})$ with magnitude 1.

Next assume that $\rho\left(T(e^{j\omega})\right) > 1$, $\forall\omega$. However, since $\lim_{\omega\to 0} T(e^{j\omega}) = T(1) \in \mathbb{R}^{p\times p}$ (the static gain) and since $\det(T(1)) \in \mathbb{R}$, it follows that $T(1)$ has a real eigenvalue with magnitude greater than 1, which contradicts (6).

Hence $\rho\left(T(e^{j\omega})\right) < 1$ must hold $\forall\omega$. Recalling that for a repeated scalar block the structured singular value is : $\mu_{\delta I_p}\left(T(e^{j\omega})\right) = \rho\left(T(e^{j\omega})\right) < 1$, leads to $\mu_{\Delta}\left(T(z)\right) < 1$, $\forall z \in \mathsf{C}$. ∎

**Remark 1** *Note that, the result in Theorem 1 is valid for arbitrary communication delays – in contrast processing delays (i.e., delays present in the system model $P(z)$ as, e.g., in example 1 in [2]) will influence the stability of the individual agents and of the MAS. Furthermore, although not influencing the stability, the communication delays will in general affect the performance of the MAS.*

### *B. Stability under Time-Varying Topology and Delays*

In the following it is implicitly assumed, that if due to the changing topology agent $k$ receives the output signals of a neighbor sent at different times, then only the most recent signal will be used, e.g., if at time $t$ an agent receives both $y_k(t-5)$ and $y_k(t-3)$ it will use only $y_k(t-3)$.

The following theorem provides a sufficient condition for stability in the case of time-varying communication topology and/or time-varying communication delays.

**Theorem 2** *The multi-agent system in Fig. 2 is stable for any number of agents $N$ and any time-varying communication topology with time-varying communication delays $\tau_{ik}(t) \in \mathbb{N}_0$, $\tau_{ik}(t) \geq 0$, $\forall t$, if there exists an invertible matrix $D \in \mathbb{R}^{p\times p}$ such that $\|DT(z)D^{-1}\|_{l_1} < 1$.*

*Proof:* Note that in the case of a communication topology with time-invariant communication delays, one can write for the error signal $\boldsymbol{e}(t)$ of the overall MAS

$$\boldsymbol{e}(t) = -\boldsymbol{y}(t) - \sum_{h=0}^{t} A_h \boldsymbol{y}(t-h), \tag{7}$$

where the matrices $A_h$ are appropriately constructed (i.e., they select the neighbors from which the agents receive information with a delay of $h$ steps).

In the case of a time-varying topology the matrices $A_h$ are also time-varying and (7) becomes

$$\boldsymbol{e}(t) = -\boldsymbol{y}(t) - \sum_{h=0}^{t} A_h(t) \boldsymbol{y}(t-h) = -\mathcal{L}_{(p)}(t)\boldsymbol{y}(t), \tag{8}$$

where $\mathcal{L}_{(p)}(t) = \mathcal{L}(t) \otimes I_p$ is a linear operator $\ell_p^\infty \to \ell_p^\infty$. By using $T(z) = (I_p + H(z))^{-1} H(z)$ and introducing the operator $\mathcal{A}_{(p)}(t) = \mathcal{L}_{(p)}(t) - I_{pN}$, one can now completely characterize the time-varying communication topology $\mathcal{A}_{(p)}(t)$ (see Fig. 3) by an infinite block lower triangular matrix $R$ (see, e.g., [22]) in the form

$$\begin{bmatrix} \boldsymbol{v}_{\mathcal{A}}(0) \\ \boldsymbol{v}_{\mathcal{A}}(1) \\ \cdots \end{bmatrix} = \underbrace{\begin{bmatrix} R_{00} & 0 & \dots & 0 \\ R_{10} & R_{11} & 0 & 0 \\ \vdots & \vdots & \ddots & \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \boldsymbol{y}(0) \\ \boldsymbol{y}(1) \\ \cdots \end{bmatrix}}_{\boldsymbol{y}}, \tag{9}$$

where $R_{th} = -A_{t-h}(t)$, and the fact is used that $\boldsymbol{w}_{\mathcal{A}}(t) = \boldsymbol{y}(t)$ (cf. Fig. 3). Note, that the matrices in row $t$ of $R$ correspond to the coefficients $A_h(t)$ in (8) and as such fully characterize the normalized adjacency matrix at time $t$. It is interesting to note that $R$ has a block-diagonal structure only in the case of a topology without communication delays, a block lower triangular structure in the case of a topology with communication delays, and a block lower triangular-band structure with band of $\bar{\tau}$ elements, if the maximal time-delay is not larger than $\bar{\tau}$ steps. However, the matrix is not Toeplitz even in the time-invariant case with time-delays, since due to the normalization of the error signal in (1) the matrix $R_{00}$ will be different than $R_{11}$, etc.

With these definitions, it is easy to see that the $l_1$ norm of $\mathcal{A}_{(p)}$ is $\|\mathcal{A}_{(p)}\|_{l_1} = \max_{\boldsymbol{y} \in \ell_p^\infty \backslash 0} \frac{\|R\boldsymbol{y}\|_\infty}{\|\boldsymbol{y}\|_\infty} = 1$ since, due to its construction based on the normalized Laplacian, the sum of the absolute values of the elements in each row of $R$ is 1 or less (1 is attained, e.g., for an undirected topology with 2 agents and no delays).

Now define a scaling matrix $\boldsymbol{D} = I_N \otimes D$, where $D \in \mathbb{R}^{p \times p}$ is invertible, and note that $\boldsymbol{D}^{-1}\mathcal{A}_{(p)}\boldsymbol{D} = \mathcal{A}_{(p)}$, since from the properties of the Kronecker product (see, e.g., [19]) $\mathcal{A}_{(p)}\boldsymbol{D} = (\mathcal{A} \otimes I_p)(I_N \otimes D) = (I_N \otimes D)(\mathcal{A} \otimes I_p) = \boldsymbol{D}\mathcal{A}_{(p)}$. Then, by applying the small-gain theorem [22, Theorem 2] to the feedback loop of $\boldsymbol{T}(z)$ and $\mathcal{A}_{(p)}$ (Fig. 3) one obtains the sufficient stability condition:

$$\begin{aligned} \|\boldsymbol{D}\boldsymbol{T}(z)\boldsymbol{D}^{-1}\|_{l_1} &= \|I_N \otimes (DT(z)D^{-1})\|_{l_1} \\ &= \|DT(z)D^{-1}\|_{l_1} < 1. \end{aligned}$$

The last equality was obtained by exploiting the fact that the $l_1$ norm of the block-diagonal system is the maximal norm over the blocks, which are identical. ■

The above theorem reduces the stability analysis of a MAS with arbitrary time-varying topology to checking an $l_1$ condition for a single agent. The following theorem shows that for a symmetric communication topology, i.e., topology satisfying $\mathcal{L}(t) = \mathcal{L}_t = \mathcal{L}_t^\mathsf{T}$, $\forall t$, without communication delays the $l_1$ norm can be replaced by the $\mathcal{H}_\infty$ norm.

**Corollary 1** *A multi-agent system is stable for any number of agents $N$ and any* symmetric *time-varying communication topology without communication delays if there exists an invertible matrix $D \in \mathbb{R}^{p \times p}$ such that $\|DT(z)D^{-1}\|_{\mathcal{H}_\infty} < 1$.*

*Proof:* Since only symmetric topologies are considered we have $\mathcal{A}_t = \mathcal{A}_t^\mathsf{T}$, $\forall t$, and hence the singular values $\sigma_i$ of $\mathcal{A}_t$ are equal to $\sqrt{\delta_i^2}$, thus satisfying $|\sigma_i| \leq 1$. Hence, $\|\mathcal{A}_t\|_{\mathcal{H}_\infty} \leq 1$, $\forall t$ and the $\mathcal{H}_\infty$ norm of $\mathcal{A}$ is not greater than one. Finally, the result

$$\|\boldsymbol{D}\boldsymbol{T}(z)\boldsymbol{D}^{-1}\|_{\mathcal{H}_\infty} = \|DT(z)D^{-1}\|_{\mathcal{H}_\infty} < 1.$$

is obtained using that $\boldsymbol{D} = I_N \otimes D$ commutes with $\mathcal{A}_{(p)} = \mathcal{A} \otimes I_p$, applying the small gain theorem (see, e.g., [20]) to the interconnection with $\boldsymbol{T}(z)$ and using that the $\mathcal{H}_\infty$ norm of a block-diagonal system is the maximal $\mathcal{H}_\infty$ norm of the systems on the diagonal. ■

Note that Theorems 1, 2 and Corollary 1 can be used to reduce the synthesis problem for a MAS with a possibly large number of agents, subject to communication delays and changing topology, to the synthesis of a controller $K(z)$ for a single agent with uncertainty. As such, the method is independent of the number of agents and the topology of a particular MAS, hence offering an efficient alternative to the design methods in [3], [6].

## IV. An Example

We illustrate the proposed design technique via an example used in [2], [8]. Each agent is a hovercraft that moves independently and with the same dynamics in x and y direction. Hence the problem can be reduced to a one-dimensional, with dynamics described by a double integrator and a one-step process delay. Because this is a SISO system, one can analyze the stability with the Nyquist stability test proposed in [2]: a MAS with fixed topology and without communication delays is stable, if and only if the Nyquist diagram of $H(e^{j\omega})$ does not encircle any of the points $-1/\lambda_i$. Since, by Lemma 1, $\lambda_i \in \{1 + \bar{\mathsf{D}}\}$, $\forall \lambda_i$, the stability under any fixed communication topology will be guaranteed if the Nyquist diagram lies to the right of the $-0.5$ line $\forall \omega$.

A robust controller for this MAS using a sensitivity shaping filter $W_S(z) = \frac{0.005}{z-0.9999}$ and $\mu$-synthesis with static scaling matrices has been designed; the Nyquist diagram of the resulting $H(z)$ is shown in Fig. 4(a). As can be seen it remains to the right of the $-0.5$ line for all frequencies and hence guarantees the stability of a MAS with arbitrary fixed topology – the attained stability value (see Theorem 1) is $\|T(z)\|_{\mathcal{H}_\infty} = 0.9992$.

A lower bound of the $l_1$ norm of $T(z)$ is computed (by considering the first 1000 Markov coefficients of this SISO system) and a value of 0.9992 is obtained, indicating that such a MAS remains stable also under any time-varying topology and communication delays. This is illustrated in Fig. 4(b) where the response of 10 hovercrafts (only in y-direction), starting from positions from 1 to 10 and with $r = 0$ as commanded input are shown, where every 10 steps a random change in the communication topology occurs and the communication delays are randomly changed to a value between 1 and 3 time steps.

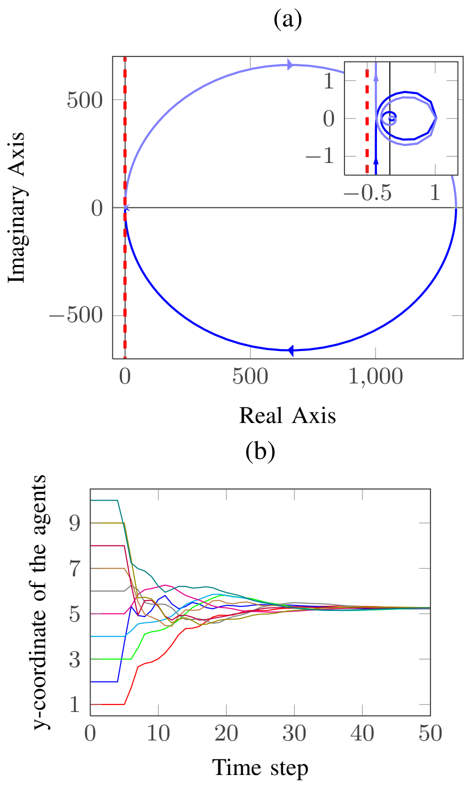


Fig. 4. For an agent with a controller designed according to Theorem 1: (a) Nyquist plot of $H(z)$ and (b) response of a MAS to $r = 0$ and subject to switching communication topology and communication delays.

## V. Conclusions

A necessary and sufficient condition for stability of a multi-agent system under arbitrary time-invariant communication topology and communication delays is proposed. In the case of a time-varying topology a sufficient scaled $l_1$ stability condition is obtained for arbitrary topologies and a scaled $\mathcal{H}_\infty$ condition for symmetric topologies without communication delays. The results are computationally attractive as they reduce the analysis and thus the synthesis task for a multi-agent system with a possibly large number of agents and changing topology to an analysis/synthesis problem for a single agent.

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