# $\mathcal{H}_\infty$ Controller Design for a Multi-Agent System Based on a Replicated Control Structure

Andrey Popov and Herbert Werner

Institute of Control Systems
Hamburg University of Technology

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TUHH
Technische Universität Hamburg-Harburg

Institut
für
Regelungstechnik

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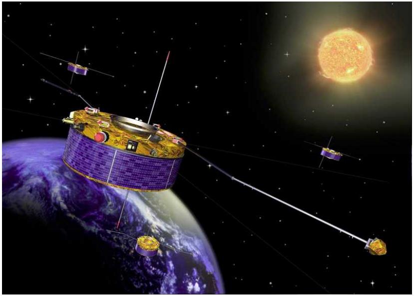
# Multi-Agent Systems



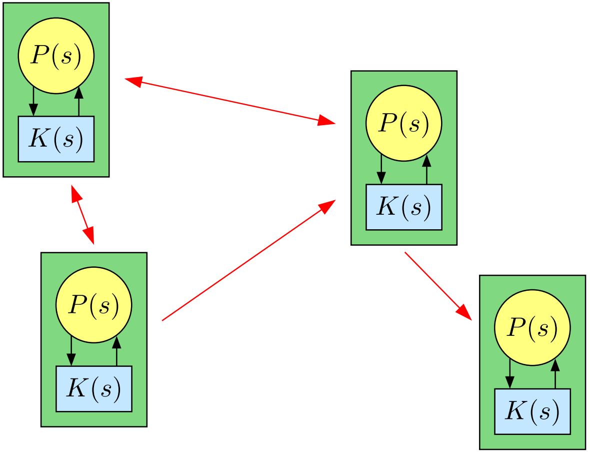
http://www.esa.int/esaSC/SEMCFF8OY2F_index_0.html

http://www.microdrones.com

## Problem formulation

For a given and fixed communication topology
design a (low-order $K \in \mathcal{K}$) controller that
when replicated to each agent results in
an optimal $\mathcal{H}_\infty$ performance of the overall multi-agent system.

# Multi-Agent System

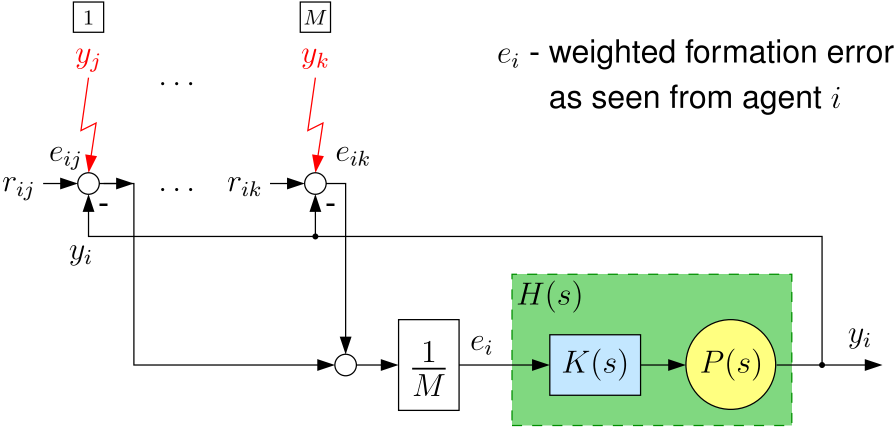



$N$ identical, LTI agents

Locally controlled

Communicating

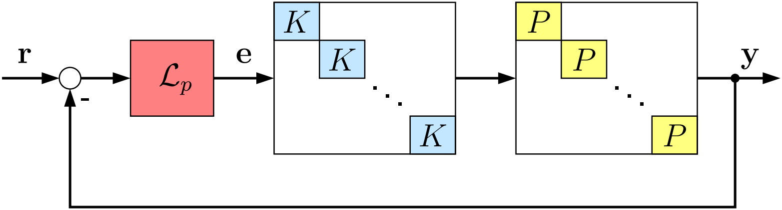
# Single Agent




$e_i$ - weighted formation error as seen from agent $i$

$$e_{ik} = r_{ik} - (y_i - y_k), \qquad e_i = \frac{1}{|\mathcal{J}_i|} \sum_{k \in \mathcal{J}_i} e_{ik}$$

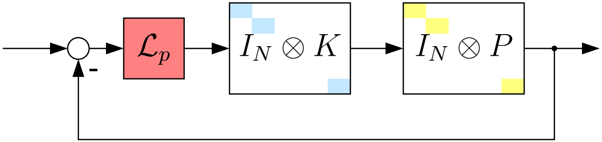
# Closed-Loop Multi-Agent System




$\mathcal{L}_p = \mathcal{L} \otimes I_p$

$\mathcal{L}$ - normalized graph Laplacian
$p$ - # communicated signals
$\otimes$ - Kronecker product

# Closed-Loop Multi-Agent System



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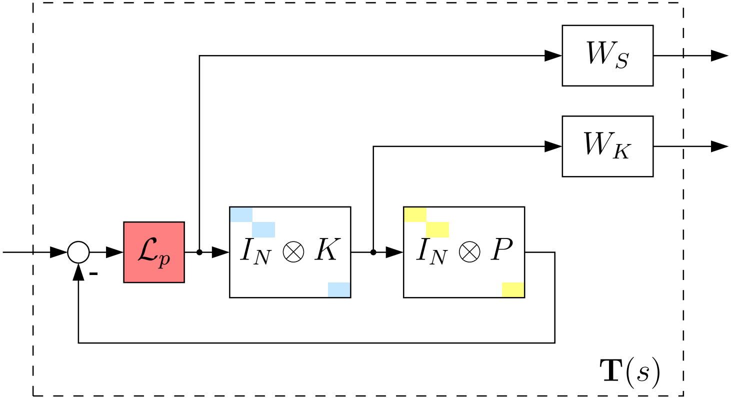
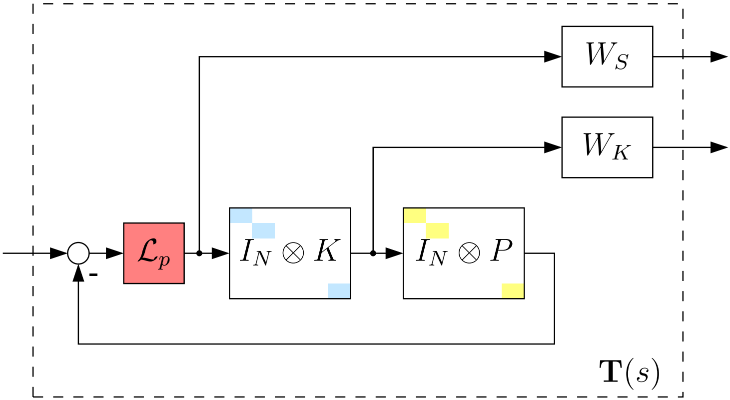


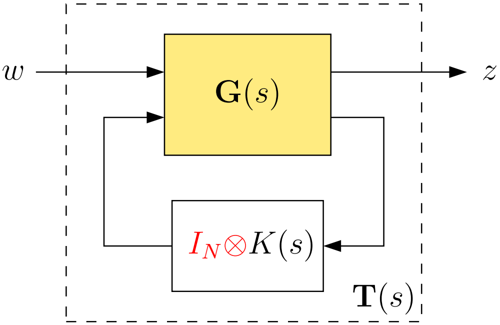

$$\underset{K(s)}{\text{minimize}} \, \|\mathbf{T}(s)\|_\infty$$

# Closed-Loop Multi-Agent System



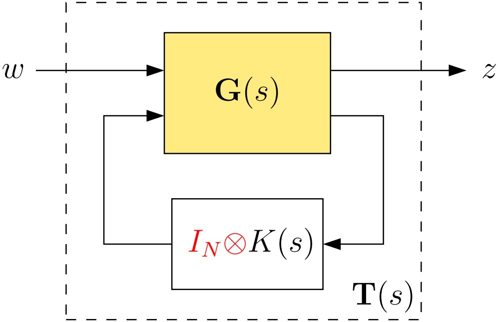

$$\underset{K(s)\in\mathcal{K}(s)}{\text{minimize}} \, \|\mathbf{T}(s)\|_\infty$$

# Closed-Loop Multi-Agent System




$$\underset{K(s)\in\mathcal{K}(s)}{\text{minimize}} \|\mathbf{T}(s)\|_\infty$$

# Closed-Loop Multi-Agent System




$$\underset{K(s)\in\mathcal{K}(s)}{\text{minimize}} \|\mathbf{T}(s)\|_\infty$$

$$\mathbf{K}(s) := I_N \otimes K(s)$$

# Contents

# Synthesis Possibilities

## LMI, Riccati

- $\mathbf{K}(s) = I_N \otimes K(s)$ cannot be imposed
- cannot impose a structure $K(s) \in \mathcal{K}(s)$

## Metaheuristic methods (EA, GA, ...)

- Computationally expensive

## Gradient optimization

- Convergence to local minimum

+ Good results with **HIFOO** (for $K \in \mathcal{K}$, but not $\mathbf{K} = I_N \otimes K$) (Gumussoy, Millstone & Overton (2008))

# Gradient-Based Synthesis Algorithm

initialize $K(s)$

1. $\mathbf{K}(s) = I_N \otimes K(s)$
2. $\mathbf{T}(s) = \mathcal{F}_L\left(\mathbf{G}(s),\, \mathbf{K}(s)\right)$
3. $f = \|\mathbf{T}(s)\|_\infty$
4. $\frac{\partial f}{\partial \mathbf{A_K}}, \dots, \frac{\partial f}{\partial \mathbf{D_K}}$ (HIFOO)
5. $\frac{\partial f}{\partial A_K}, \dots, \frac{\partial f}{\partial D_K}$ ?
6. If gradients $\neq 0$ update $A_K, \dots, D_K$ and go to 1.

# Computation of $\frac{\partial f}{\partial A_K}$

- $\frac{\partial f}{\partial \mathbf{A_K}}$ – available $\qquad f = \|\mathbf{T}(s)\|_\infty$
- $\mathbf{A_K} = I_N \otimes A_K = \sum_h \mathcal{P}_h \otimes a_h, \qquad a = \text{vec}(A_K)$

## Theorem

Given

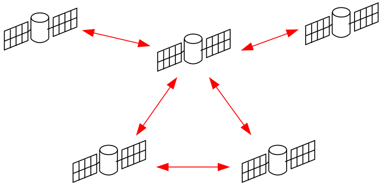
$$\mathbf{A_K}(a) = \sum_r \mathcal{P}_h \otimes a_h \ : \ \mathbb{R}^r \to \mathbb{R}^{n\times n}$$

$$f(\mathbf{A_K}) \ : \ \mathbb{R}^{n\times n} \to \mathbb{R}$$

Then

$$\frac{\partial f}{\partial a_h} = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{P}_{h_{ij}} \frac{\partial f}{\partial \mathbf{A_K}_{ij}}$$

# Example: Satellite Constellation



Satellite model: $4^{\text{th}}$ order
Generalized plant: $40^{\text{th}}$ order

| Controller | order | $\mathcal{H}_\infty$ perf. |
|---|---|---|
| Centralized | 40 | 1.02 |
| Replicated | 4 | 1.22 |

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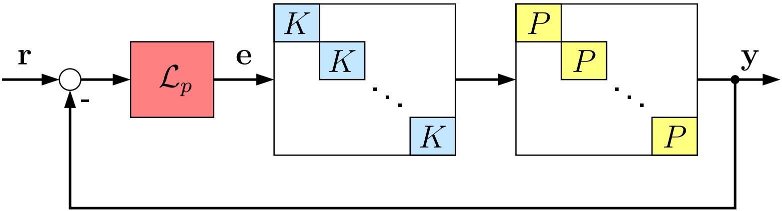
# Closed-Loop Multi-Agent System

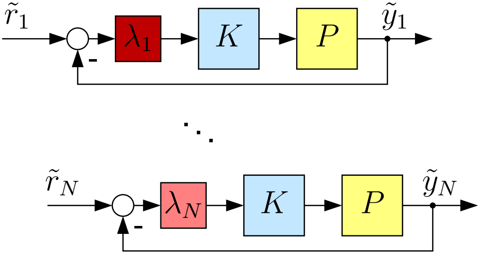



$\mathcal{L}_p = \mathcal{L} \otimes I_p$

$\mathcal{L}$ - normalized graph Laplacian

transform with $Q$ : $\mathcal{L} = QUQ^{-1}$

# Stability Analysis

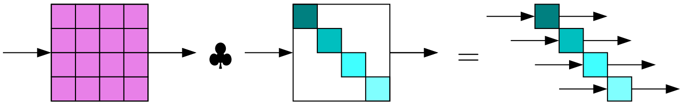



**Theorem (Fax & Murray (2003))**

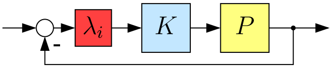
*For a given topology $\mathcal{L}$ a MAS is stable*
*if and only if*
*the $N$ closed-loop transfer functions above are stable.*

$\lambda_i \in \mathbb{C}$ - eigenvalues of $\mathcal{L}$

# Problem - Performance Guarantees




Eigenvalue diagonalization



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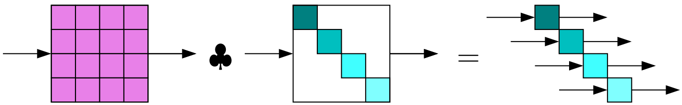


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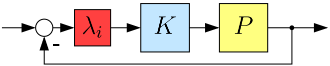


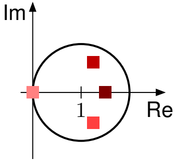

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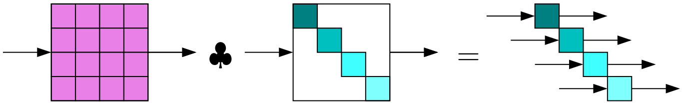


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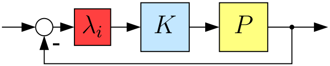
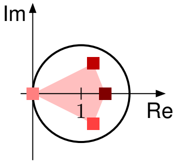
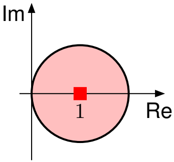



Decomposition

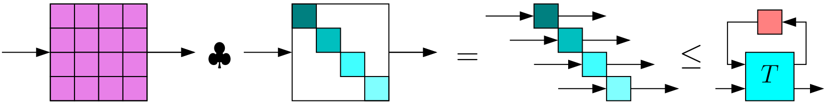
Massioni & Verhaegen




Robust

Popov & Werner

# Problem - Performance Guarantees

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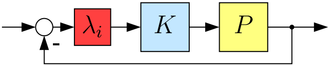


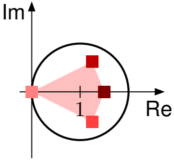


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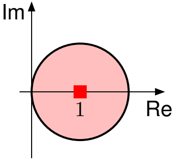


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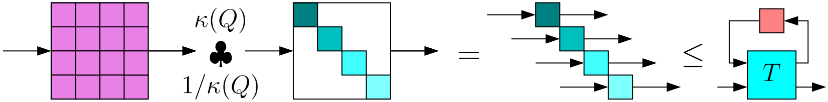
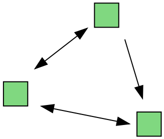
# Problem - Performance Guarantees

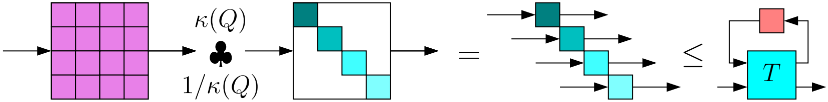



Eigenvalue diagonalization

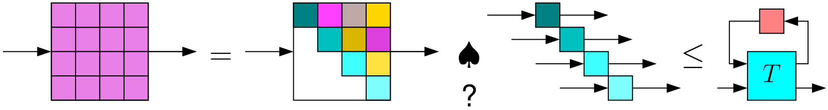


$$\kappa(Q) = \infty$$

# Problem - Performance Guarantees

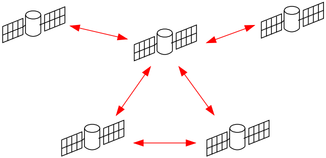



Eigenvalue diagonalization




Schur transformation

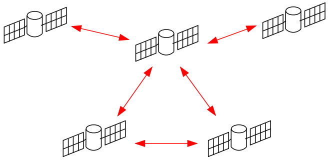
# Example: Satellite Constellation - Continued



$$\kappa(Q) = 1.95$$

| Controller | Order | $\mathcal{H}_\infty$ norm |
|---|---|---|
| Centralized | 40 | 1.02 |
| | | |
| Replicated | 4 | 1.22 |

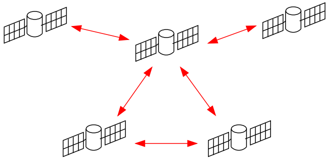
# Example: Satellite Constellation - Continued



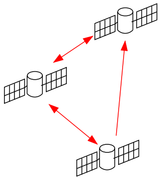
$$\kappa(Q) = 1.95$$

| Controller | Order | $\mathcal{H}_\infty$ norm |
|---|---|---|
| Centralized | 40 | 1.02 |
| Decomp. (1.02) | 8 | 1.21 |
| Replicated | 8 | 1.17 |
| | 4 | 1.22 |

# Example: Satellite Constellation - Continued



$$\kappa(Q) = 1.95$$



$$\kappa(Q) = \infty$$

| Controller | Order | $\mathcal{H}_\infty$ norm |
|---|---|---|
| Centralized | 40 | 1.02 |
| Decomp. (1.02) | 8 | 1.21 |
| Replicated | 8 | 1.17 |
| | 4 | 1.22 |

| Controller | Order | $\mathcal{H}_\infty$ norm |
|---|---|---|
| Centralized | 40 | 1.01 |
| Decomp. (1.02) | 8 | 1.30 |
| Replicated | 8 | 1.06 |
| | 4 | 2.14 |

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# Conclusions

**Method for $\mathcal{H}_\infty$ synthesis of**

- controllers with replicated elements
- fixed-structure agent controllers

**The method is applicable to**

- general MIMO systems
- any fixed communication topology

and

- provides performance guarantees for multi-agent systems

# Conclusions

**Method for $\mathcal{H}_\infty$ synthesis of**

- controllers with replicated elements
- fixed-structure agent controllers

**The method is applicable to**

- general MIMO systems
- any fixed communication topology

and

- provides performance guarantees for multi-agent systems

**However,**

- suboptimal - convergence to a local minima
- applicable to systems with moderate size

Thank you for your attention!

Questions?